SPRING 2025 MATH 590: QUIZ 6

Name:

1. State the theorem that characterizes when an $n \times n$ matrix A s diagonalizable over F. Be sure to define all terms used in the statement of the theorem. (5 points)

Solution. The matrix A is diagonalizable if and only if the following conditions hold: (i) $p_a(x)$ has all of its roots in F, i.e., $p_A(x) = (x-\lambda_1)^{e_1}\cdots(x-\lambda_r)^{e_r}$, for $\lambda_i \in F$ and $e_i \geq 1$ and (ii) For each $1 \leq i \leq r$, $\dim(E_{\lambda_i}) = e_i$, i.e., for each eigenvalue λ_i, the geometric multiplicity of λ_i equals the algebraic multiplicity of λ_i.

Terms: $p_a(x) = |xI_n - A|$ or $p_A(x) = |\lambda I_n - A|$ - the order doesn't matter, either one can be defined as $p_A(x)$.

E_{λ_i} is the eigenspace of λ_i.

Use of the terms geometric and algebraic multiplicity is not required, but if used: the geometric multiplicity of λ_i is the dimension of E_{λ_i} and the algebraic multiplicity of λ_i is e_i.

2. Diagonalize the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, and verify that eigenvectors corresponding to the two distinct eigenvalues are orthogonal. This is an important property of symmetric matrices. (5 points)

Solution. $p_A(x) = \begin{vmatrix} x-1 & -2 \\ -2 & x-1 \end{vmatrix} = (x-1)^2 - 4 = x^2 - 2x - 3 = (x+1)(x-3)$, so the eigenvalues of A are -1 and 3.

$E_{-1} =$ null space of $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a basis for E_{-1}.

$E_3 =$ null space of $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. Thus, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for E_3.

The diagonalizing matrix is $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, and $P^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$,

$P^{-1}AP = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -1 & 1 \\ -3 & -3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

Taking the dot product, $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1+1 = 0$, therefore, the corresponding eigenvectors are orthogonal.